A GEOMETRIC VIEW ON THE WILSONIAN RENORMALIZATION GROUP

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Geometrization of effective actions

- Can one provide a more conceptual underpinning for the AdS/CFT correspondence a la Wilson?

- Necessary to understand how to generalize holographic ideas.
- Crucial to gain insight into how quantum gravity works.

- Provide an understanding for origin of local physics in the bulk.
- Allow identification of the effective degrees of freedom and their interactions for intrinsically strongly coupled systems.

OUTLINE

- Motivation
- Geometrizing the Wilsonian RG
- Example 1: Scalar fields
- Semi-holography via RG
- Example 2: Vectors
- Looking ahead

The Scale Radius duality

- One entry in the AdS/CFT dictionary:
  - Energy scale of the field theory maps to the radial coordinate of AdS.
  - $\rightarrow$ leads to the UV/IR duality
  - Boundary of AdS: UV of field theory
  - Interior of AdS: IR region of field theory

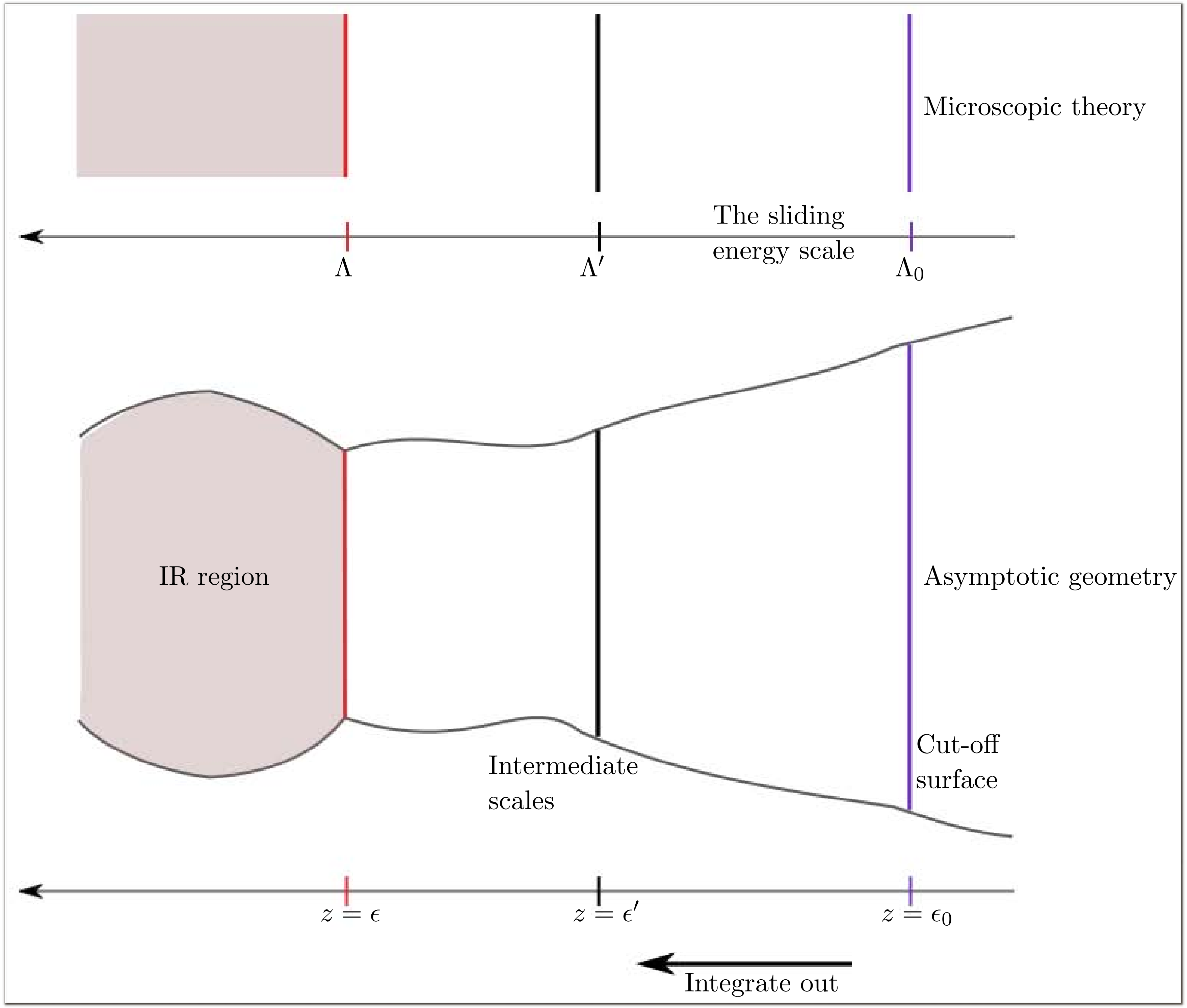
Susskind, Witten

- Naively: Integrating out energy shells in the boundary field theory should therefore map to integrating out a part of the bulk geometry.
- This is at the heart of the Holographic Renormalization Group.

de Boer, Verlinde, Verlinde

- We are going to attempt to do something similar and derive a flow equation, albeit with some differences.

A conceptual picture



A few caveats

- Any given process in the field theory involves excitation at all scales in the bulk:
  - clear for example in the fluid/gravity correspondence, where one describes the low energy effective theory in terms of an asymptotically AdS spacetime.
  - $\exists$ some limitations to the UV/IR duality: relativistic beaming — Hubeny
- The map between the cut-off scale in field theory and the bulk radial position is not terribly transparent
  - this is key: we need to unravel this map to understand bulk locality.
  - e.g., make precise the statement that a cut-off bulk AdS is dual to CFT coupled to induced gravity
  - multi-trace operators play an important role

Geometric RG: the proposal

- Consider an effective field theory defined by a path integral:

$$Z = \int_\Lambda D\Phi \, \exp\left[i\, I_{eff}[\Phi,\Lambda]\right] \qquad I_{eff}[\Phi,\Lambda] = I_0[\Phi] + I_{UV}[\Phi,\Lambda]$$

microscopic action

result of integrating out

- The action I_{UV} obeys an renormalization group flow equation.

- In the dual gravity theory we consider a bulk action:

$$S = \int_{z>\epsilon} d^{d+1}x\, \sqrt{-g}\, \mathcal{L}(\phi, \partial_M \phi) + S_B[\phi, \epsilon]$$

specifies boundary state & is bulk analog of I_{UV}

The proposal

- The bulk boundary term S_B is identified with I_{UV}
  - directly when we think of alternate quantization of boundary CFT which is allowed in a small range of conformal dimensions of dual operator.
  - up to a Legendre transformation when we work with the standard quantization.

- Care should be exercised in interpreting S_B:
  - all regions of the bulk contribute to a given physical process.
  - non-local terms induced in S_B due to gapless modes in the geometry.

- We'll derive a flow equation for $S_B[\varepsilon]$
  - the flow is driven by the 'bulk Hamiltonian'
  - equation is the WKB limit of the equation proposed in Heemskerk, Polchinski

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Example 1: Scalars

- Let us consider a bulk scalar field described by an action

$$S = \int_{z>\epsilon} d^{d+1}x \sqrt{-g}\,\mathcal{L}(\phi, \partial_M \phi) + S_B[\phi, \epsilon]$$

- The cut-off surface $z = \varepsilon$ being arbitrary, the on-shell action should be unchanged as we move the surface:

$$0 = -\int_{z=\epsilon} d^d x \sqrt{-g}\,\mathcal{L} + \partial_\epsilon S_B[\phi, \epsilon] + \int_{z=\epsilon} d^d x \frac{\delta S_B}{\delta \phi(x)} \partial_z \phi(x).$$

- This can be rewritten in terms of a Hamiltonian flow

$$\partial_\epsilon S_B[\phi, \epsilon] = -\int_{z=\epsilon} d^d x \left(\Pi \partial_z \phi - \sqrt{-g}\,\mathcal{L}\right) = -\int d^d x\, \mathcal{H}$$

Some salient points

- The flow equation for S_B is a functional equation and provides a simple way to encode the evolution of the couplings.
- Important that one not impose the bulk equations of motion in evaluating this flow equation.

- Thus $S_B[\varepsilon]$ only encodes information about the part of the geometry that has been integrated out.
- It has no information about the interior of the geometry.
- This is the main difference from the viewpoint of the Holographic Renormalization group.

de Boer, Verlinde, Verlinde

- We will also see momentarily that $S_B[\varepsilon]$ contains multi-trace operators in addition to single-trace even in the planar limit.
- Makes this quantity conceptually different from the counter-term actions written down for holographic renormalization.

Skenderis, Papadimitriou,....

Flow equations for the scalar

- To see some more important features consider a free scalar in the bulk.

$$S_B[\epsilon,\phi] \;=\; \Lambda(\epsilon) + \int \frac{d^dk}{(2\pi)^d}\sqrt{-\gamma}\,J(k,\epsilon)\phi(-k) - \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\sqrt{-\gamma}\,f(k,\epsilon)\,\phi(k)\phi(-k)$$

- The flow equations for the 'couplings' in S_B are:

$$\mathcal{D}_\epsilon\Lambda = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\,J(k,\epsilon)J(-k,\epsilon),$$

$$\mathcal{D}_\epsilon\left(\sqrt{-\gamma}\;J(k,\epsilon)\right) = -J(k,\epsilon)\;f(k,\epsilon),$$

evolution of the source

$$\mathcal{D}_\epsilon\left(\sqrt{-\gamma}\;f(\epsilon,k)\right) = -f^2(k,\epsilon) + k^\mu k_\mu + m^2$$

evolution of double trace couplings

$$\mathcal{D}_\epsilon = \frac{1}{\sqrt{-g}}\,\partial_\epsilon$$

A note on conventions

- Bulk metric:

$$ds^2 = g_{MN}dx^M dx^N \equiv -g_{tt}dt^2 + g_{ii}d\vec{x}^2 + g_{zz}dz^2$$

- On the hypersurface of interest:

$$\gamma_{\mu\nu}\, dx^\mu\, dx^\nu = -g_{tt}(\epsilon)dt^2 + g_{ii}(\epsilon)d\vec{x}^2$$

$$\sqrt{-g} = \sqrt{-\gamma\, g_{zz}}$$

- Momenta conventions:

$$\begin{aligned} k_\mu &= (-\omega, k_i), & d^d k &= d\omega\, d^{d-1}k_i, \\ k^2 &\equiv \sum_i k_i^2, & k^\mu k_\mu &= -g^{tt}\omega^2 + g^{ii}k^2 \end{aligned}$$

Interpreting flow equations

- The flow equations are simply related to the classical equations of motion.
- One can check that f and J can be mapped to a classical solution via:

$$f = -\frac{\Pi_c}{\sqrt{-\gamma}\phi_c}, \qquad J = \frac{1}{\sqrt{-\gamma}\phi_c}$$

- As such there is no simplification if we want to solve for the evolution of the couplings for all values of momenta.
- However, as in the case of RG, these equations are immensely helpful in extracting the low energy behaviour.

- The evolution equation for f was derived as a bulk extension of a boundary two-point function to understand the change in transport coefficients between the boundary and a bulk black hole horizon.

Iqbal, Liu

Double-trace flow

- To appreciate the evolution of f consider pure AdS$_{d+1}$:

$$\epsilon\,\partial_\epsilon f = -f^2 - \Delta\Delta_- + d\,f \qquad \Delta = \frac{d}{2}+\nu, \qquad \nu = \sqrt{\frac{d^2}{4}+m^2}, \qquad \Delta_- = d-\Delta$$

- In terms of $f = \bar{f} + \Delta_-$ one derives the beta-function equation for double trace couplings

$$\epsilon\,\partial_\epsilon \bar{f} = -\bar{f}^2 + 2\nu\bar{f}$$

- $\bar{f}$ is the renormalized double-trace coupling. In the continuum limit either:
  - CFT with alternative quantization (Δ_-) perturbed by a double-trace
  - Standard quantization with double trace deformation

$$W[\mathcal{O}_-] = \int\left(J_-\mathcal{O}_- - \frac{1}{2}\kappa_-\mathcal{O}_-^2\right) \qquad W_+[\mathcal{O}_+] = \int\left(J_+\mathcal{O}_+ - \frac{1}{2}\kappa_+\mathcal{O}_+^2\right)$$

$$\bar{f}(\epsilon) = \kappa_-\epsilon^{2\nu}, \qquad J(\epsilon) = J_-\epsilon^{\Delta}, \qquad J_+ = -\frac{J_-}{\kappa_-}, \qquad \kappa_+ = -\frac{1}{\kappa_-}\;.$$

Double-trace flow

- The fixed point of the double-trace flow correspond to the standard and alternative quantization respectively.
- Standard quantization is the IR fixed point
- Alternative quantization is the UV fixed point

$$\bar{f}(\epsilon) = \frac{\kappa_- \epsilon^{2\nu}}{1 + \frac{\kappa_-}{2\nu}\epsilon^{2\nu}} \,, \qquad J(\epsilon) = \frac{J_- \epsilon^{\Delta}}{1 + \frac{\kappa_-}{2\nu}\epsilon^{2\nu}}$$

- The double trace operators are generated along the flow even if we start from the undeformed theory.
- This fact plays an important role in understanding effective description of CFTs at non-zero density and zero temperature.

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Semi-holographic models

- Investigations of retarded Green's functions of probe fields in extremal black hole backgrounds have revealed interesting behaviour: Gubser
  - scalars tend to want to condense in the near horizon Hartnoll, Herzog, Horowitz
  - fermionic Green's functions reveal characteristics of Fermi surface, with non-Fermi liquid behaviour Faulkner, Liu, McGreevy, Vegh

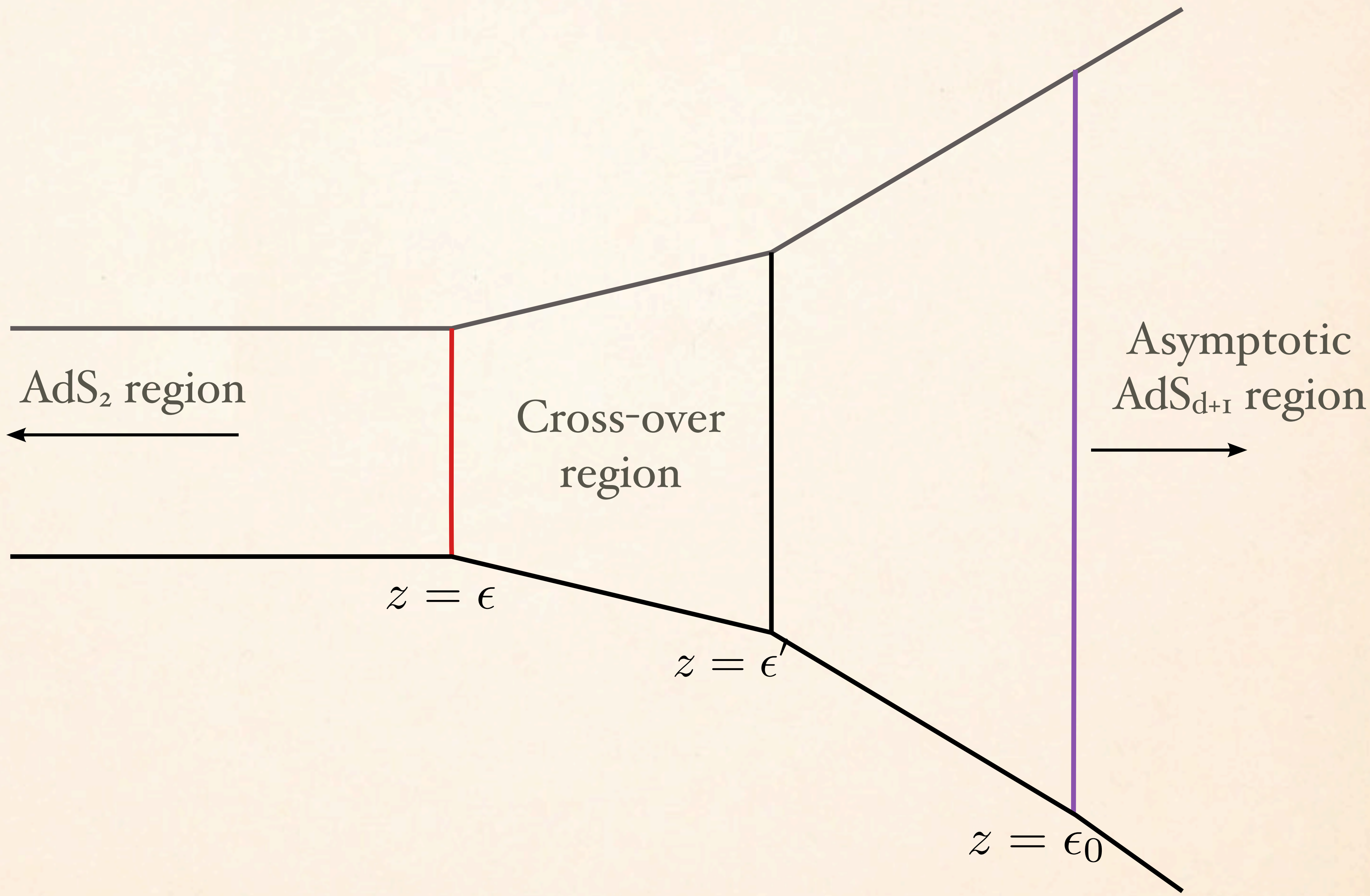
- In all cases the interesting physics seems to be due to the infinite throat of extremal black holes, resulting in an AdS_2 geometry in the near horizon.

Kunduri, Lucietti, Reall

- The effective description of these systems can be captured in terms of semi-holographic models.
- These models take seriously the AdS_2 part of the spacetime and imagine the CFT degrees of freedom coupled to near-horizon modes.

Faulkner, Polchinski

Semi-holographic models



Semi-holographic models

- By analyzing the behaviour of double-trace operators one can derive the semi-holographic description from the effective action S_B.
- Integrate the flow equation & relate the double-trace deformation of the UV/boundary theory to that of the IR/AdS_2 theory.

$$\kappa_I = -\frac{b_+ - \kappa_U\, a_+}{b_- - \kappa_U\, a_-}$$

- For standard quantization of the UV theory, and the IR physics is governed by the coefficients $a_\pm$ since $\kappa_U \to \infty$.
- Scalar instability, Fermi surfaces, etc., are related to zeros of a_+ which occur at some special values of momenta.
- In the vicinity of these points in momentum space,

$$\kappa_I = c\,\omega^2 + \cdots$$

Semi-holographic models

- The effective action for AdS$_2$ is rendered non-local $\frac{1}{2}\int \frac{1}{c\,\omega^2+\cdots}\,\Psi_+^2$
- Locality maybe regained by:
- working in terms of the alternative quantization (if admissible) $\frac{1}{2}\int \kappa_I \Psi_-^2$
- realizing that there is a gapless mode in the spacetime and retain this mode in the effective action:

$$S = \underline{S_0(z>\epsilon) + S_{ct}(z=\epsilon)} + \int_{z=\epsilon} \frac{1}{2}\kappa_I\,\phi_0^2$$

$$\left\langle e^{\int \phi_0\,\Psi_+} \right\rangle$$

- The gapless mode ensures locality of the low energy effective action. One simply has

$$\frac{1}{2}\int c_2\,(\partial_t\phi_0)^2 + \int \phi_0\Psi_+$$

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Example 2: Vectors

- Let us consider a bulk Maxwell field with action

$$S = S_0[z > \epsilon, A_M] + S_B[A_M, \epsilon]$$

- The flow equation:

$$\partial_\epsilon S_B[A_\mu, \epsilon] = -\int_{z=\epsilon} d^d x \sqrt{-g} \left[\frac{1}{2\gamma}\, g_{\mu\nu} \frac{\delta S_B}{\delta A_\mu} \frac{\delta S_B}{\delta A_\nu} + \frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \right] + \int d^d x\, \partial_\mu \frac{\delta S_B}{\delta A_\mu} A_z \ .$$

- In order to understand the flow we need to sort out issues relating to gauge symmetries. Two possibilities:

◆ Dirichlet boundary condition for the Maxwell field

◆ Neumann boundary condition (admissible in *d=3* and *d=4*).

Vectors with Dirichlet bc

- The Dirichlet bc is the conventional bc for vectors in AdS/CFT at the boundary. We simply fix the boundary value of the gauge field

$$A_\mu(z=0,x) = B_\mu(x)$$

- Perform the bulk path integral integrating out modes living between boundary and the cut-off surface.

$$e^{iS_B[A_M,\epsilon]} = \int_{\tilde{A}_\mu(z=0,x)=B_\mu(x)}^{\tilde{A}_M(z=\epsilon,x)=A_M(x)} [D\tilde{A}_M]\, e^{iS_0[\tilde{A}_M]}$$

- Choose radial gauge and incorporate the Goldstone mode

Nickel, Son

$$\hat{A}_\mu \equiv A_\mu - \partial_\mu \varphi, \qquad \varphi(x) = \int_0^\epsilon dz\, \tilde{A}_z$$

- These gauge invariant fields are useful to parameterize the boundary action S_B

Flow equations for vectors

- One can write down a general form of the boundary action

$$
\begin{aligned}
S_B[\hat{A}_\mu, \epsilon] &= \Lambda(\epsilon) + \int \frac{d^d k}{(2\pi)^d} \sqrt{-\gamma} \left(J^\mu(k,\epsilon)\, \hat{A}_\mu(-k) - \frac{1}{2} f_T(k,\epsilon)\, A_i^T(k) A^i_T(-k) \right) \\
&\quad - \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \sqrt{-\gamma} \left(f_0(k,\epsilon) \hat{A}_0(k) \hat{A}^0(-k) + f_L(k,\epsilon) g^{ii} \hat{A}^L(k) \hat{A}^L(-k) \right) \\
&\quad - \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \sqrt{-\gamma} \left[f_{0L} (\hat{A}_0(k) \hat{A}^L(-k) + \hat{A}_0(-k) \hat{A}^L(k)) \right]
\end{aligned}
$$

- Working in a translationally invariant boundary geometry (such as a planar black hole), the equations separate into longitudinal and transverse sectors

$$
\mathcal{D}_\epsilon \left(\sqrt{-\gamma}\, J^i_T(k,\epsilon) \right) = -J^i_T(k,\epsilon)\, f_T(k,\epsilon)
$$

$$
\mathcal{D}_\epsilon \left(g^{ii} \sqrt{-\gamma}\, f_T(k,\epsilon) \right) = g^{ii} \left(-f_T^2(k,\epsilon) + k_\mu k^\mu \right)
$$

$$
\mathcal{D}_\epsilon (\sqrt{-\gamma} J^0) = -J^0 f_0 - g_{ii} J^L f_{0L}
$$

$$
\mathcal{D}_\epsilon (\sqrt{-\gamma} J^L) = -J^L f_L + g_{tt} J^0 f_{0L}
$$

$$
\begin{aligned}
\mathcal{D}_\epsilon (\sqrt{-\gamma} g^{tt} f_0) &= -g^{tt} f_0^2 + g_{ii} f_{0L}^2 + g^{tt} g^{ii} k^2 \\
\mathcal{D}_\epsilon (\sqrt{-\gamma} g^{ii} f_L) &= -g^{ii} f_L^2 + g_{tt} f_{0L}^2 - g^{tt} g^{ii} \omega^2 \\
\mathcal{D}_\epsilon (\sqrt{-\gamma} f_{0L}) &= f_{0L}(f_L + f_0) - g^{tt} g^{ii} \omega k
\end{aligned}
$$

$$
\mathcal{D}_\epsilon \Lambda = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} J^\mu(k,\epsilon)\, J_\mu(-k,\epsilon)
$$

Flow equations for vectors

- It is interesting to examine the low frequency behaviour of the couplings.
- Explicit solutions for the couplings can be obtained in terms of known functions of metric functions Q_t and Q_i

$$S_B[\hat{A}_\mu, \epsilon] = \int \left[\frac{1}{2\,Q_t} \left(\hat{A}_0 - B_0 \right)^2 - \sum_i \frac{1}{2\,Q_i} \left(\hat{A}_i - B_i \right)^2 \right]$$

- Legendre transformation of this boundary actions gives us the CFT effective action

$$I_{UV} = \int \left(-\frac{1}{2} Q_t (j^0)^2 + \frac{1}{2} \sum_i Q_i (j^i)^2 - j^\mu (B_\mu + \partial_\mu \varphi) \right)$$

- Note that the Goldstone mode's presence in S_B keeps the action local. If we integrate it out, we would end up with a non-local action

$$\tilde{S}_B[A_\mu, \epsilon] = -\int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{2} \frac{1}{Q_i \omega^2 - Q_t k^2} (E^L)^2 + \sum_i \frac{1}{2Q_i} (A_i^T)^2 \right]$$

Diffusion on the horizon

- From the effective action for the vector fields we can derive the diffusion equation for a conserved current in a black hole background.

- Effective action + in-falling boundary condition on the horizon leads to diffusive modes with dispersion:

$$\omega = -iD\, k^2, \qquad D = \sigma\, Q_t$$

- This derivation of diffusion from the effective action by integrating out modes all the way down to the horizon is a clean way to derive the black hole membrane paradigm.
- The result here complements the previous analysis of understanding how to relate the boundary fluid dynamics to the effective action on the stretched horizon.

Iqbal, Liu

Bredberg, Keeler, Lysov, Strominger

Nickel, Son

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Summary

- A general prescription for a Wilsonian approach to the holographic renormalization group.
- We explicitly integrate out regions of the bulk spacetime and use it to induce an effective action on the cut-off surface.

- At the level of classical gravity and for probe fields (scalars, vectors):
- Obtain the beta-function equations for multi-trace operators
- Make contact with the semi-holographic models.
- Provide a clean derivation of diffusion on the stretched horizon a la the black hole membrane paradigm.

Open issues

- What about dynamical gravity?

- would like to see an effective action for the hydrodynamic modes which arise from integrating out the degrees of freedom between the boundary and the horizon.
- construct effective actions for low energy modes in extremal black hole backgrounds.

- What is the role of alternative quantization?

- seems to provide a simpler way to understand the process of integrating out the geometry.
- retains the light degrees of freedom.
- perhaps a useful tool to understand induced gravity?